

2.9. Linear Approximation.

Key point: Linearization of f at $a =$ Tangent line of f at a .

Formula: $L(x) = f(a) + f'(a) \cdot (x-a)$. (*)

Goal/Motivation: How to estimate $\sqrt{9.01}$ without calculator? More precisely, intuitively, $\sqrt{9.01} \approx \sqrt{9} = 3$, we want to get a more accurate approximation via $L(x)$.

Method: Pick the suitable function $f(x)$ and a , Apply linearization Formula $L(x)$ as an approximation of the desired value of $f(x)$.

eg.1. Consider $f(x) = \sqrt{x}$. Find its linearization at $x=9$.

Rank: It is equivalent to ask "Find the tangent line at $x=9$ ".

solution: Apply formula (*) directly.

$$a=9, f(a) = \sqrt{9} = 3, f'(x) = \frac{1}{2} x^{\frac{1}{2}-1} = \frac{1}{2} x^{-\frac{1}{2}} \Rightarrow \begin{cases} f'(9) = \frac{1}{2} \cdot 9^{-\frac{1}{2}} \\ = \frac{1}{2} \cdot \frac{1}{\sqrt{9}} = \frac{1}{6} \end{cases}$$

therefore, Linearization of $f(x)$ at 9 is:

$$L(x) = 3 + \frac{1}{6} \cdot (x-9)$$

eg.2. Use a linearization to find a good approximation of $\sqrt{9.01}$.
(F16). Hint: Consider the function $f(x) = \sqrt{x}$ and point $x=9$ in eg.1.

$$\text{eg.1} \Rightarrow L(x) = 3 + \frac{1}{6} \cdot (x-9) \quad (\text{linearization})$$

Plug in 9.01 (since we want to estimate $\sqrt{9.01}$)

$$L(9.01) = 3 + \frac{1}{6} (9.01 - 9) = 3 + \frac{1}{6} \cdot 0.01 = \boxed{3 + \frac{1}{600}}$$

eg 3 If $f(1)=3$ and $f'(1)=5$, use linear approximation to estimate $f(0.99)$ (sib, MC). Solution: $a=1$, $f(1)=3$, $f'(1)=5$.

$$\Rightarrow \text{Linearization: } L(x) = f(1) + f'(1) \cdot (x-1) = 3 + 5 \cdot (x-1).$$

$$\begin{aligned} \text{To estimate } f(0.99), \text{ plug in } x=0.99, \quad L(0.99) &= 3 + 5 \cdot (0.99-1) = 3 + 5 \cdot (-0.01) \\ &= 3 - 0.05 = \boxed{2.95}. \end{aligned}$$

• Error in the linear approximation.

$$L(x) - f(x) = f'(a) \cdot (x-a), \text{ i.e., Derivative } f'(a) \text{ times the error of the variable.}$$

eg 4. We want to measure the radius of a sphere to calculate its surface area (Formula: $A = 4\pi \cdot r^2$). The radius is measured to be 5 cm with possible error of 0.1 cm. What's the maximum error in the calculated surface area?

$$\text{Solution: } A(r) = 4\pi \cdot r^2. \quad a=5. \quad A'(r) = 4\pi \cdot 2r$$

(Caution: this is not Related Rates Prob. The derivative is w.r.t. r)

$$A(5) = 4\pi \cdot 25 = 100\pi, \quad A'(5) = 4\pi \cdot 2 \cdot 5 = 40\pi.$$

$$\text{The linearization of } A(r) \text{ at } 5 \text{ is: } 100\pi + 40\pi \cdot (r-5)$$

The error in the estimate is

$$f'(a)(x-a), \text{ i.e., } 40\pi \cdot (r-5).$$

Now the error for radius is at most 0.1, i.e., $r-5$ is at most 0.1

$$\text{Therefore, the error for the surface area is } 40\pi \cdot 0.1 = 4\pi$$

§3.1 Extreme Values

Key points: ① Absolute (Global) maximum/minimum.

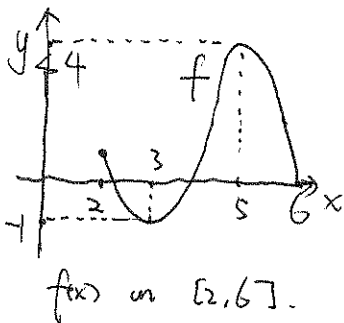
② Critical points (numbers)

③ Local maximum/minimum. (More details in 3.3, 3.5)

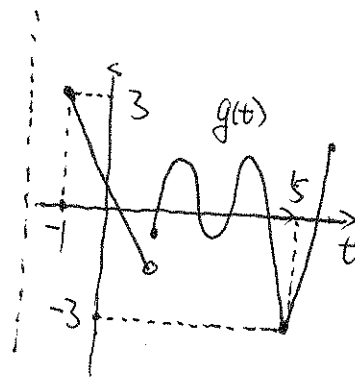
Goal: want to find the maximum/minimum values of $f(x)$ through its derivative $f'(x)$.

• Def: For $f(x)$ defined on $[a, b]$. If there is c in $[a, b]$ such that $f(c) \geq f(x)$ for all x in $[a, b]$, then we say $f(x)$ has **ABSOLUTE MAXIMUM VALUE** $f(c)$ at $x=c$. (Or, the absolute maximum occurs at $x=c$.)

Absolute minimum is defined similarly with $f(c) \leq f(x)$.



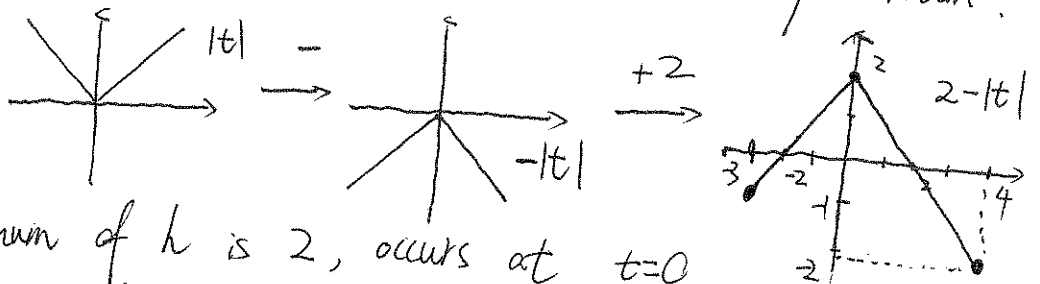
$f(x)$ attains its absolute maximum $f(5)=4$ at 5, and its absolute minimum $f(3)=-1$ at 3.



$g(t)$ attains its absolute maximum $g(-1)=3$ at $t=-1$ and its absolute minimum $g(5)=-3$ at $t=5$.

eg. 1. $h(t) = 2 - |t|$ on $[-3, 4]$. Find its absolute maximum/minimum.

Solution: Draw the graph of $2 - |t|$.



$h(t)$: absolute maximum of h is 2, occurs at $t=0$
absolute minimum of h is -2, occurs at $t=4$

• Def: Critical points (numbers) are all the points x such that $f'(x)=0$ or $f'(x)$ is not defined ($f'(x)$ D.N.E / $f(x)$ is not differentiable).

- Method to find all critical points for $f(x)$ on $[a, b]$.

Step 1: Take derivative. (Compute $f'(x)$).

Step 2: Set the denominator of $f'(x)$ to be zero and solve for x . (where $f'(x)$ is undefined).

Step 3: Set $f'(x) = 0$. Solve for x . (where $f'(x)$ is zero).

Step 4: Discard those points not in $[a, b]$. All the rest points in Steps 2, 3 are your answers. (critical points)

eg. 2. Let $f(t) = -t + 4\sqrt{t}$. Find all critical numbers of f in $[0, \frac{25}{5}]$

Solution: S1: $f'(t) = (-t + 4\sqrt{t})' = (-t)' + (4 \cdot t^{\frac{1}{2}})' = -1 + 4 \cdot \frac{1}{2} \cdot t^{-\frac{1}{2}} = -1 + 2 \cdot t^{-\frac{1}{2}}$

S2: Now $f'(t) = -1 + 2 \cdot \frac{1}{\sqrt{t}}$. (\sqrt{t} is in the denominator, $\Rightarrow \sqrt{t}$ can't be zero.)
i.e. $\sqrt{t} = 0 \Rightarrow t = 0$ ($t = 0$ is where $f'(t)$ is undefined)

S3: Set $f'(t) = 0 \Rightarrow -1 + 2 \cdot \frac{1}{\sqrt{t}} = 0 \Rightarrow \frac{2}{\sqrt{t}} = 1 \Rightarrow \sqrt{t} = 2 \Rightarrow t = 4$

S4: Both 0, 4 are in $[0, \frac{25}{5}]$. f has critical numbers $t = 0, t = 4$ in $[0, \frac{25}{5}]$.

eg. 3 Find all the critical points for $f(x) = 48x - x^3$ over $[-5, 2]$

Solution: $f'(x) = (48x - x^3)' = 48 - 3x^2$

(No step 2 needed. No denominator in $f'(x)$. i.e. $f'(x)$ is defined for all x).

$f'(x) = 0 \Rightarrow 48 - 3x^2 \Rightarrow 48 = 3x^2 \Rightarrow x^2 = 16 \Rightarrow x = \pm 4$.

therefore, f has critical point $x = -4$ in $[-5, 2]$. ($x = 4$ is not in $[-5, 2]$)

Remark: If we change the interval in eg. 3 to be $[-1, 2]$, then

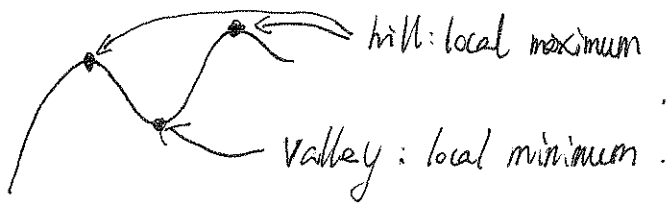
$f(x) = 48x - x^3$ has NO CRITICAL POINTS IN $[-1, 2]$.

Neither $x = 4$ or $x = -4$ is in $[-1, 2]$

- Def: For $c \in (a, b)$, if $f(c)$ is the largest (smallest) value for all $f(x)$ near $x = c$, then $f(c)$ is a local maximum (minimum) of $f(x)$.

Remark: local maximum is a point where the graph has a hill

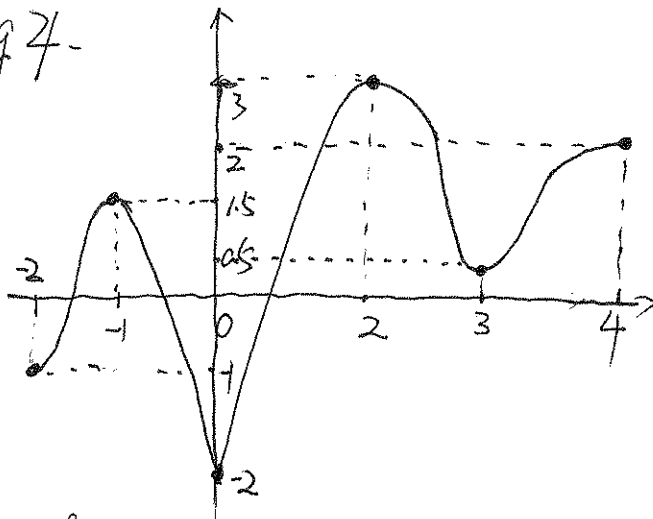
local minimum is a point where the graph has a valley



Absolute maximum is the highest hill.

Absolute ~~valley~~ ^{minimum} is the lowest valley.

eg 4.



$f(x)$ on $[-2, 4]$

• (Hills) local maximum:

$$f(-1) = 1.5, f(2) = 3, f(4) = 2.$$

• Absolute maximum is the largest among the above, i.e., $f(2) = 3$

• (Valleys) local minimum:

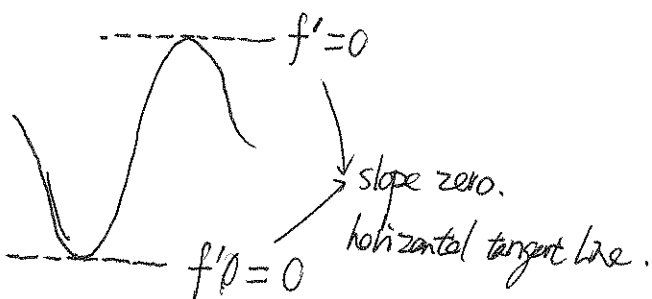
$$f(-2) = -1, f(0) = -2, f(3) = 1.5$$

• Absolute minimum is the smallest among the above, i.e., $f(0) = -2$.

Remark (Theorem): If $f(x)$ has a local extremum (maximum/minimum)

at $x=c$ in (a,b) , and $f'(c)$ exists, then $f'(c) = 0$.

In the graph, it means $f(x)$ has a horizontal tangent line (with slope zero) at these local extrema.



In eg 4, $f'(-1) = 0$, $f'(2) = 0$, $f'(3) = 0$

$f'(0)$ is local minimum (and absolute minimum)

but $f'(0)$ does not exist.

★ Method to find absolute maximum/minimum of $f(x)$ on $[a, b]$

Step 1: Find all CRITICAL POINTS of f in $[a, b]$.

Step 2: List all values of f at the above critical points, $f(c)$.
List $f(a)$, $f(b)$. (the values of f at endpoints)

Step 3: Compare all values listed in Step 2.

The largest is the absolute maximum. The smallest is the absolute minimum.

eg. 5. Find the absolute maximum and minimum for $f(t) = -t + 4\sqrt{t}$ in $[0, 25]$ considered in eg. 2.

Solution: Step 1: Critical points: $t=0$, $t=4$.

Step 2: f at critical points: $f(0) = 0$, $f(4) = -4 + 4\sqrt{4} = 4$.

f at ~~the~~ endpoints: $f(0) = 0$, $f(25) = -25 + 4\sqrt{25} = -5$

Step 3: $f(0) = 0$, $\underbrace{f(4) = 4}_{\text{largest}}$, $\underbrace{f(25) = -5}_{\text{smallest}}$.

f has absolute maximum $f(4) = 4$, occurs at $t = 4$

f has absolute minimum $f(25) = -5$, occurs at $t = 25$.

eg 6. Find the absolute maximum value of $f(x) = 48x - x^3$ over $[-1, 2]$

(SIB, MC) $f'(x) = 48 - 3x^2 = 0 \Rightarrow x^2 = 16 \Rightarrow x = \pm 4$ not in $[-1, 2]$.

$f(x)$ has no critical points in $[-1, 2]$. (See Remark after eg 3).

(List endpoints only): $f(-1) = 48(-1) - (-1)^3 = -48 + 1 = -47$
 $x = -1, x = 2$.

$$f(2) = 48 \cdot 2 - 2^3 = 96 - 8 = 88$$

Therefore, the absolute maximum of f over $[-1, 2]$

is $\boxed{88}$, which occurs at $x = 2$.